REMARKS. The numerical work was done on a Stardent 1500 computer. A FORTRAN code of about 300 lines was constructed for the computations described in this paper; in addition, routines from LINPACK were used to solve the linear systems. A copy of the program is available upon request from the author.

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Rational Interpolation via Orthogonal Polynomials

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Abstract—Relations between rational interpolants and Hankel matrices are investigated. A modification of a Jacobi-like algorithm for rational interpolation proposed in [1] is developed. The proposed algorithm eliminates all the auxiliary assumptions required to implement the algorithm of [1]. In this way, it can be used to construct all existing rational interpolants with the cost of $O(N^2)$ arithmetic operations where $N+1$ is the number of data points.

1. INTRODUCTION

Let $R(m,n)$ be the set of rational functions $r_{m,n}(x) = p_m(x)/q_n(x)$ where $p_m(x)$ and $q_n(x)$ are polynomials of degree m and n, respectively. The problem of rational interpolation is that of finding $r_{m,n}(x) \in R(m,n)$, which assumes given values $f_0, \dots, f_N$ at distinct points $x_0, \dots, x_N$, where $N = m+n$.

If $n = 0$, the problem is reduced to polynomial interpolation and the solution exists and is unique.

For fixed m and n, a direct algorithm for rational interpolation consists of finding the solution of

$$p_m(x_i) - f_i q_n(x_i) = 0, \qquad 0 \le i \le N. \tag{1}$$

Equations (1) are a set of $N+1$ homogeneous linear equations in $N+2$ unknowns, and a nontrivial solution always exists. However, if $p_m(x_i) = q_n(x_i) = 0$ for a certain x_i, $0 \le i \le N$, then a solution of (1) is not necessarily a solution of the rational interpolation problem. The points (x_i, f_i) for which

$$p_m(x_i) = f_i q_n(x_i), \qquad r_{m,n}(x_i) := \lim_{x \to x_i} \frac{p_m(x_i)}{q_n(x_i)} \ne f_i$$

are called the unattainable points, and (x_i, f_i), $0 \le i \le N$, becomes a degenerate configuration for the pair (m,n).

A fast version of the classical Jacobi's algorithm for rational approximation has been proposed in [1]. It is based on the computation of the denominator polynomial $q_n(x)$,

$$q_n(x) = \sum_{i=0}^{n} b_i x^i, \qquad b_n = 1, \tag{2}$$

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by means of the following matrix equation

$$\begin{pmatrix} h_0 & h_1 & h_2 & \dots \\ h_1 & h_2 & \dots & \dots \\ \dots & \dots & \dots & \dots \\ h_{n-1} & h_n & \dots & h_{2n-2} \end{pmatrix} \begin{pmatrix} b_0 \\ \vdots \\ \vdots \\ b_{n-1} \end{pmatrix} = - \begin{pmatrix} h_n \\ \vdots \\ \vdots \\ h_{2n-1} \end{pmatrix}, \tag{3}$$

where $h_j = \sum_{k=0}^{N} \frac{f_k x_k^j}{w_k}$, $w_k = \prod_{j=0, j\neq k}^{N} (x_k - x_i)$.

Relation (3) is quite intriguing by the computational complexity viewpoint. The matrix H_{n-1} on the left-hand side of equation (3) is a Hankel matrix. Therefore, to determine the coefficients of $q_n(x)$, one has to solve a Hankel system, and if one wants to find $q_n(x)$ for all n, then this amounts to solving a nested sequence of Hankel matrices. Due to the Hankel structure of the system (3), it is possible to solve it with stable methods having a cost complexity of $O(N^2)$ arithmetic operations. Algorithms for solving Hankel systems with this sequential complexity are referred to as fast algorithms [2]. Recently, superfast $O(n \log^2 n)$ algorithms have appeared [2–4]. Both the fast and superfast algorithms rely upon the solution of smaller subproblems and they are expressed in terms of submatrices or bordering techniques. Since most of these algorithms can be shown to be unstable, they either explicitly assume exact arithmetic or implicitly do so by not considering the effects of rounding errors. However, all these methods require the condition of strong nonsingularity of the Hankel matrix (i.e., all leading principal minors are nonzero).

This additional assumption turns out to be equivalent to the existence of a sequence $q_0(x)$, $q_1(x), \dots, q_n(x)$ of orthogonal polynomials on the discrete set $\{x_0, \dots, x_n\}$ with respect to the symmetric bilinear form on the space Π_n of polynomials of degree less than or equal to n,

$$\langle t_j(x), t_k(x) \rangle = \sum_{i=0}^{N} \frac{f_i}{w_i} t_j(x_i) t_k(x_i). \tag{4}$$

Orthogonal polynomials obey the well-known three-term recurrence relation

$$q_{j+1}(x) = (x - \alpha_j) q_j(x) - \beta_j q_{j-1}(x), \qquad q_{-1}(x) = 0,\ q_0(x) = 1, \tag{5}$$

where α_j and β_j are constants determined by

$$\alpha_j = \frac{\langle x q_j(x), q_j(x) \rangle}{\langle q_j(x), q_j(x) \rangle}, \qquad \beta_j = \frac{\langle q_j(x), q_j(x) \rangle}{\langle q_{j-1}(x), q_{j-1}(x) \rangle}. \tag{6}$$

Now, if $f_i \neq 0$, $0 \leq i \leq N$, we may compute the coefficients of the polynomial $p_m(x)$ of the rational interpolant $r_{m,n}(x)$ by applying the same technique to the data set (x_i, f_i^{-1}), $0 \leq i \leq N$, provided that the strong nonsingularity of the matrix H'_{m-1}

$$H'_{m-1} = \begin{pmatrix} h'_0 & h'_1 & h'_2 & \dots \\ h'_1 & h'_2 & \dots & \dots \\ \dots & \dots & \dots & \dots \\ h'_{m-1} & h'_m & \dots & h'_{2m-2}, \end{pmatrix}, \tag{7}$$

where $h'_j = \sum_{k=0}^{N} \frac{x_k^j}{f_k w_k}$.

In this way, we arrive at the following direct algorithm for rational interpolation [1].

ALGORITHM 1.

INPUT. (x_i, f_i), $f_i \neq 0$, $0 \leq i \leq N$.

OUTPUT. *Coefficients of the polynomials* $p_m(x)$ *and* $q_n(x)$, $0 \leq n \leq N$, $m = N - n$ *of the rational interpolants* $r_{m,n}(x)$.

(1) Compute $q_n(x_j)$, $0 \le j \le N$, $0 \le n \le N$, and the coefficients of $q_n(x)$ by means of the three-term recurrence relations (5) and (6).
(2) Check if $q_n(x_j) = 0$, $0 \le j \le N$, that is, check whether or not the data set is a degenerate configuration.
(3) Compute $\tilde{p}_m(x_j)$, $0 \le j \le N$, $m = N - n$, by performing Step 1 on the data set (x_i, f_i^{-1}), $0 \le i \le N$.
(4) Set $p_m(x) = \alpha\tilde{p}_m(x)$, where α is such that $q_n(x_j)/\tilde{p}_m(x_j)\, f_j = \alpha$ for (x_j, f_j) attainable point.

Here and hereafter, we denote as $\tilde{p}(x)$ the monic polynomial obtained by normalizing to unity the first nonzero coefficient of $p(x)$.

Reference [1] proves the following result.

PROPOSITION 1. *If $H_0, H_1, \dots, H_{N-1}$ and $H'_0, H'_1, \dots, H'_{N-1}$ are nonsingular, then Algorithm 1 computes all rational interpolants $r_{m,n}(x)$, $0 \le n \le N$, $m = N - n$, using $O(N^2)$ arithmetic operations.*

Surely, all the $N(N+1)$ coefficients of the polynomials $p_m(x)$ and $q_{N-m}(x)$ such that $r_{m,N-m}(x) = p_m(x)/q_{N-m}(x)$ cannot be computed with less than $O(N^2)$ arithmetic operation, that is Algorithm 1 is optimal with respect to the sequential complexity.

Our contribution is to eliminate the hypotheses of Proposition 1 by looking carefully at the general case where breakdowns occur in the three-term recurrence relation (5) or, equivalently, either H_{N-1} or H'_{N-1} have some singular leading principal submatrix.

The main tools for our investigation are the properties of the Euclidean Algorithm. The study of interpolation problem using the Euclidean Algorithm dates back to Kronecker [5], and, more recently, Meinguet [6] and Antoulas [7]. Motivated by (3), we will concentrate our attention on the relations between the Euclidean Algorithm and the (block) triangular factorization of the Hankel matrices (see [8,9]), by showing that Algorithm 1 performs a triangular factorization of the matrices H_N and H'_N.

By means of a classical theorem due to Gragg and Lindquist [10], in Section 2, we relate the breakdown occurrence in the matrices H_{N-1} and H'_{N-1} by establishing the following result

$$\det H_{j-1} \neq 0 \quad \Longleftrightarrow \quad \det H'_{N-j} \neq 0, \quad 1 \le j \le N. \tag{8}$$

Relation (8) shows, among other things, that the strong nonsingularity of H_{N-1} implies the strong nonsingularity of H'_{N-1} and vice-versa.

In Section 3, we discuss a suitable modification of the above algorithm, called Algorithm 2, which deals with the case where $\langle q_j(x), q_j(x)\rangle = 0$ for a certain j by jumping from every nonsingular leading principal submatrix to the next one. Similar techniques have already been developed in different contexts by Luenberger [10] to extend the conjugate gradient algorithm, by Bunch, Kauffman and Parlett [11] to stabilize triangular factorization, and by Parlett [12] and Freund, Gutknecht and Nachtigal [13] to analyze the Lanczos algorithm.

Further, we show that Algorithm 2 preserves the cost complexity of Algorithm 1. In this way, the proposed algorithm solves the rational interpolation problem with the lowest sequential cost without no restrictions on the problem being solved.

2. ANALYSIS OF BREAKDOWN OCCURRENCE

Given the data set (x_i, f_i), $f_i \neq 0$, $0 \le i \le N$ and $x_i \neq x_j$ if $i \neq j$, then let $v(x) \in \Pi_N$ and $z(x) \in \Pi_N$ be the unique interpolating polynomials such that $v(x_i) = f_i$ and $z(x_i) = f_i^{-1}$, $0 \le i \le N$. By setting $w(x) = \Pi_{i=0}^{N}(x - x_i)$, $w'(x_k) = w_k$, it can easily be observed that

$$z(x)v(x) \equiv 1 \bmod (w(x)); \tag{9}$$

that is, $\gcd(z(x), w(x)) = \gcd(v(x), w(x)) = 1$.

The Hankel matrices H_N and H'_N can be factorized as $H_N = V^T D_v V$ and $H'_N = V^T D_z V$, where V is the Vandermonde matrix $V = (x_i^j)_{i,j=0,N}$ and

$$D_v = \begin{pmatrix} \frac{v(x_0)}{w'(x_0)} & & \\ & \ddots & \\ & & \frac{v(x_N)}{w'(x_N)} \end{pmatrix}, \qquad D_z = \begin{pmatrix} \frac{z(x_0)}{w'(x_0)} & & \\ & \ddots & \\ & & \frac{z(x_N)}{w'(x_N)} \end{pmatrix}. \tag{10}$$

Incidentally, we now observe that these factorizations imply the positive definiteness of the Hankel matrices H_N and H'_N under the conditions

$$f_i f_{i+1} < 0, \qquad i = 0, \dots, N-1; \tag{11}$$

that is, (11) defines a class of problems which Algorithm 1 can be applied to. The nonsingularity of all the matrices H_0, $H_1, \dots,$ H_{N-1} turns out to be equivalent to the nondegeneracy of the bilinear form (4). Moreover, under this assumption, the Euclidean Algorithm applied to the polynomials $r_{N+1}(x) = w(x)$ and $r_N(x) = \tilde{z}(x)$

$$r_{j+1}(x) = (x - a_j) r_j(x) - b_j r_{j-1}(x), \qquad j = N, N-1, \dots, \tag{12}$$

generates a set of monic orthogonal polynomials with respect to the bilinear form (4) [7,14]; that is, we have $r_j(x) = q_j(x)$, $a_j = \alpha_j$ for $0 \le j \le N$ and $b_j = \beta_j$ for $1 \le j \le N$.

By stating relation (12) in matrix form, we obtain

$$r_j(x) = \det(xI - T_j), \qquad 1 \le j \le N+1,$$

where T_j is the left principal submatrix of order j of the tridiagonal matrix

$$T_{N+1} = \begin{pmatrix} \alpha_0 & \beta_1 & & & & \\ 1 & \alpha_1 & \beta_2 & & & \\ & \ddots & \ddots & \ddots & & \\ & & \ddots & \ddots & \ddots & \\ & & & 1 & \alpha_{N-1} & \beta_N \\ & & & & 1 & \alpha_N \end{pmatrix}.$$

Let S_j be the permutation matrix of order j having unit antidiagonal entries. Correspondingly, set

$$\bar{r}_j(x) = \det(xI - (S_{N+1} T_{N+1} S_{N+1})_j), \qquad \bar{r}_0(x) = 1,\ \bar{r}_{-1}(x) = 0.$$

We have [14].

LEMMA 1. *For $0 \le j \le N$ $r_N(x_j)\bar{r}_N(x_j) = \beta_1 \dots \beta_N$.*

Since this lemma implies that $\bar{r}_N(x)$ and $z(x)$ coincide up to a scalar multiple, we have the following result.

PROPOSITION 2. *H_N is strongly nonsingular if, and only if, H'_N is strongly nonsingular.*

The next objective to arise, after looking at Proposition 2, is to link the occurrence of breakdowns in the matrices H_N and H'_N. For this purpose, we need the following result [10].

PROPOSITION 3. *Let $0 = m_0 < m_1 < \dots < m_L = N+1$ be integers such that $\det H_{m_i - 1} \ne 0$, $i = 1, \dots, L$; otherwise $\det H_k = 0$. Set $\delta_i = m_i - m_{i-1}$. Let $\mathcal{D}_N$ be the class of block diagonal matrices $\mathrm{diag}(D_1, \dots, D_L)$ where D_i is a $\delta_i \times \delta_i$ lower triangular (with respect to the antidiagonal)*

Hankel matrix and Z the down-shift matrix having entries $z_{i,j} = 1$ if $i = j+1$; otherwise, $z_{i,j} = 0$. Then we have the following results.

- *There exists an upper triangular matrix R having unit diagonal entries, such that $R^T H R = D$, $D \in \mathcal{D}_N$. In particular, one can choose*

$$R = \tilde{R} = (\mathbf{q}_0, Z\mathbf{q}_0, \dots, Z^{\delta_1 - 1}\mathbf{q}_0, \mathbf{q}_1, Z\mathbf{q}_1, \dots, Z^{\delta_2-1}\mathbf{q}_1, \dots, Z^{\delta_L - 1}\mathbf{q}_{L-1}),$$

  where $\mathbf{q}_i^T = (\mathbf{s}_i^T, 1, \mathbf{0}^T)$, $\mathbf{s}_i = -H_{m_i-1}^{-1}(h_{m_i}, \dots, h_{2m_i-1})^T$, $i = 1, \dots, L$.
- *Any upper triangular matrix $R = (\mathbf{r}_1, \dots, \mathbf{r}_{N+1})$ such that $R^T H R \in \mathcal{D}_N$ is such that $\mathbf{r}_{m_i+1} = \mathbf{q}_i$, $0 \le i \le L-1$.*
- *Set $Q_i(x) = (1, x, \dots, x^N)\mathbf{q}_i$, $0 \le i \le L-1$; then, we have*

$$Q_{-1}(x) = 0, \qquad Q_0(x) = 1,$$

$$Q_i(x) = c_i(x)\, Q_{i-1}(x) - \theta_{i-1}\, Q_{i-2}(x),$$

  where $c_i(x)$, $i = 0, \dots, L$ are monic polynomials of degree δ_i.

Moreover,

$$F\tilde{R} = \tilde{R}T,$$

where F is the companion matrix associated to $Q_L(x) = \sum_{i=0}^{N} q_i x^i + x^{N+1}$

$$F = \begin{pmatrix} 0 & 0 & \dots & -q_0 \\ 1 & 0 & \dots & -q_1 \\ 0 & \ddots & \ddots & \vdots \\ 0 & 0 & 1 & -q_N \end{pmatrix},$$

and T is the $(N+1) \times (N+1)$ block tridiagonal matrix

$$T = \begin{pmatrix} T_{1,1} & T_{1,2} & & & & \\ T_{2,1} & T_{2,2} & T_{2,3} & & & \\ & \ddots & \ddots & \ddots & & \\ & & \ddots & \ddots & \ddots & \\ & & & T_{L-1,L-2} & T_{L-1,L-1} & T_{L-1,L} \\ & & & & T_{L,L-1} & T_{L,L} \end{pmatrix},$$

where $T_{k,k}$ is the $\delta_k \times \delta_k$ companion matrix associated to $c_k(x)$ and $T_{k+1,k}$ and $T_{k,k+1}$ are matrices of size $\delta_{k+1} \times \delta_k$ and $\delta_k \times \delta_{k+1}$, respectively, given by

$$T_{k+1,k} = \begin{pmatrix} & 1 \\ \bigcirc & \end{pmatrix}, \qquad T_{k,k+1} = \begin{pmatrix} & \theta_k \\ \bigcirc & \end{pmatrix}.$$

Applying Proposition 3 to the matrix H_N, we have the following [9].

PROPOSITION 4. *Let $H_N = (R^T)^{-1} D R^{-1}$, $D \in \mathcal{D}_N$ be a block triangular factorization of H_N defined as in Proposition 3. Then, we have*

$$Q_L(x) = w(x), \qquad Q_{L-1}(x) = \tilde{z}(x),$$

that is, the monic polynomials $Q_i(x)$ are generated by the Euclidean Algorithm applied to the polynomials $w(x)$ and $\tilde{z}(x)$.

Now let us consider the tridiagonal matrix,

$$\tilde{T} = S_{N+1} T S_{N+1} = \begin{pmatrix} \tilde{T}_{1,1} & \tilde{T}_{1,2} & & & & \\ \tilde{T}_{2,1} & \tilde{T}_{2,2} & \tilde{T}_{2,3} & & & \\ & \ddots & \ddots & \ddots & & \\ & & \ddots & \ddots & \ddots & \\ & & & \tilde{T}_{L-1,L-2} & \tilde{T}_{L-1,L-1} & \tilde{T}_{L-1,L} \\ & & & & \tilde{T}_{L,L-1} & \tilde{T}_{L,L} \end{pmatrix},$$

where $\tilde{T}_{j,j} = S_{\delta_{L-j+1}} T_{L-j+1} S_{\delta_{L-j+1}}$, $\tilde{T}_{j,j+1} = S_{\delta_{L-j+1}} T_{L-j+1,L-j} S_{\delta_{L-j}}$ and $\tilde{T}_{j+1,j} = S_{\delta_{L-j}} T_{L-j,L-j+1} S_{\delta_{L-j+1}}$.

Let T' and $\tilde{T}'$ be the left principal submatrices of T and $\tilde{T}$ of order $\sum_{i=1}^{L-1} \delta_i = m_{L-1}$ and $\sum_{i=2}^{L} \delta_i = N + 1 - m_1$, respectively. Let T'' be the principal submatrix of T which has entries in the last $\sum_{i=2}^{L} \delta_i$ rows and columns of T.

It is obvious that $\det(xI - \tilde{T}) = w(x)$ and $\det(xI - \tilde{T}') = \det(xI - T'')$.

Since T is nonderogatory, i.e., all its eigenvalues have unit geometric multiplicity, there exist two vectors which coincide up to a scalar multiple

$$\mathbf{u} = \begin{pmatrix} \mathbf{u}^{(1)} \\ \mathbf{u}^{(2)} \\ \vdots \\ \mathbf{u}^{(L)} \end{pmatrix}, \qquad \tilde{\mathbf{u}} = \begin{pmatrix} \tilde{\mathbf{u}}^{(1)} \\ \tilde{\mathbf{u}}^{(2)} \\ \vdots \\ \tilde{\mathbf{u}}^{(L)} \end{pmatrix},$$

such that

$$u_{\delta_1}^{(1)} = \tilde{u}_{\delta_L}^{(L)} = 1, \qquad (x_i I - T)\mathbf{u} = (x_i I - \tilde{T})\tilde{\mathbf{u}} = \mathbf{0}.$$

Then, by applying Cramer's rule to T' and T'', we obtain

$$\tilde{u}_{\delta_1}^{(1)} = \frac{\theta_1 \theta_2 \dots \theta_{L-1}}{\det(x_i I - T')}, \qquad \tilde{u}_{\delta_L}^{(L)} = \frac{\theta_1 \theta_2 \dots \theta_{L-1}}{\det(x_i I - T'')}.$$

From $\tilde{u}_{\delta_1}^{(1)} u_{\delta_L}^{(L)} = 1$, we get

$$\det(x_i I - T') \det(x_i I - T'') = \theta_1 \theta_2 \dots \theta_{L-1}, \qquad 0 \le i \le N;$$

this implies

$$\det(xI - \tilde{T}') = \det(xI - T'') = \tilde{v}(x).$$

Therefore, we have proved the main result of this section.

PROPOSITION 5. *For the given Hankel matrices, H_N and H_N', the following relation holds.*

$$\det H_{j-1} \neq 0 \iff \det H'_{N-j} \neq 0 \qquad 1 \le j \le N.$$

Proposition 5 implies that the matrices H_N and H_N' define a parametrization of all solutions of rational interpolation problem with respect to a given degree of complexity (compare [7]).

3. A BREAKDOWN-INSENSITIVE ALGORITHM

In this section, we propose a suitable modification of Algorithm 1 for the solution of the rational interpolation problem in order to overcome the occurrence of breakdown in the triangular factorization of H_N and H_N'.

Let us return to the block triangular factorization of $H_N = (\tilde{R}^T)^{-1} D \tilde{R}^{-1}$, $D \in \mathcal{D}_N$,

$$D = \begin{pmatrix} D_1 & & \\ & \ddots & \\ & & D_L \end{pmatrix}, \qquad D_i = \begin{pmatrix} & & & & d_1^{(i)} \\ & & & d_1^{(i)} & d_2^{(i)} \\ & & \ddots & \ddots & \vdots \\ d_1^{(i)} & d_2^{(i)} & \dots & & d_{\delta_i}^{(i)} \end{pmatrix}, \tag{13}$$

as has been pointed out in Proposition 3.

Combining relations (10) and (13), we obtain

$$\tilde{R}^T V^T D_v V \tilde{R} = D. \tag{14}$$

Since

$$V\tilde{R} = \begin{pmatrix} Q_0(x_0) & x_0 Q_0(x_0) & \dots & x_0^{\delta_1 - 1} Q_0(x_0) & \dots & x_0^{\delta_L - 1} Q_{L-1}(x_0) \\ Q_0(x_1) & x_1 Q_0(x_1) & \dots & x_1^{\delta_1 - 1} Q_0(x_1) & \dots & x_1^{\delta_L - 1} Q_{L-1}(x_1) \\ \dots & \dots & \dots & \dots & \dots & \dots \\ Q_0(x_N) & x_0 Q_0(x_N) & \dots & x_N^{\delta_1 - 1} Q_0(x_N) & \dots & x_N^{\delta_L - 1} Q_{L-1}(x_N) \end{pmatrix},$$

then relation (14) states an orthogonality property of the sequence of polynomials $Q_0(x)$, $xQ_0(x)$, $\dots, x^{\delta_1-1}Q_0(x), \dots, x^{\delta_L-1}Q_{L-1}(x)$ with respect to the, eventually, degenerate symmetric bilinear form (4) on the space Π_N.

PROPOSITION 6. *For the sequence of polynomials $Q_0(x)$, $xQ_0(x), \dots, x^{\delta_1-1}Q_0(x), \dots x^{\delta_L-1}$ $Q_{L-1}(x)$, we have*

$$\langle x^k Q_j(x), x^h Q_i(x) \rangle = \begin{cases} 0, & \text{if } i \neq j; \\ 0, & \text{if } i = j,\ k + h < \delta_{i+1} - 1; \\ d_{h+k+2-\delta_{i+1}}^{(i+1)}, & \text{elsewhere;} \end{cases}$$

where $0 \le k \le \delta_{j+1} - 1$, $0 \le h \le \delta_{i+1} - 1$ and $0 \le j, i \le L$.

In principle, Proposition 6 allows us to extend Algorithm 1 to the general case when breakdowns in the three-term recurrence relation (5) occur. Below, we describe the resulting method where we introduce the function

$$pr(j) = \max\{i < j : \exists h \text{ such that } i = m_h\},\ pr(1) = 0,\ pr(0) = -1,$$

where $\det H_{m_i - 1} \neq 0$, $i = 1, \dots, L$; otherwise $\det H_k = 0$; and $q_0(x) = 1$, $q_{-1}(x) = 0$.

ALGORITHM 2.

INPUT. (x_i, f_i), $f_i \neq 0$, $0 \le i \le N$.

OUTPUT. Coefficients of the denominator polynomial $q_n(x)$, $0 \le n$, $j \le N$, $\det H_{n-1} \neq 0$, of the rational interpolant $r_{m,n}(x)$.

1. Initialize $j = 0$;
2. If $\langle q_j(x), q_j(x) \rangle \neq 0$, then
   - Compute the coefficients of $q_{j+1}(x)$ and the values $q_{j+1}(x_i)$ by means of

     $$q_{j+1}(x) = (x - \alpha) q_j(x) - \beta q_{pr(j)}(x),$$

     where

     $$\alpha = \frac{\langle x q_j(x), q_j(x) \rangle}{\langle q_j(x), q_j(x) \rangle},\ \beta = 0 \text{ if } j = 0, \text{ otherwise } \beta = \frac{\langle x q_j(x), q_{j-1}(x) \rangle}{\langle q_{pr(j)}(x), q_{j-1}(x) \rangle}.$$

   - Set $pr(j+1) = j$, $j = j + 1$;

3. If $\langle q_j(x), q_j(x)\rangle = 0$, then
   - Compute $k \geq j$ such that $\langle q_j(x), q_k(x)\rangle \neq 0$ setting $q_{j+h}(x) = x^h q_j(x)$, $1 \leq h \leq k-j$;
   - compute the coefficients of $q_{k+1}(x)$ and the values $q_{k+1}(x_i)$ by means of

   $$q_{k+1}(x) = c(x)q_j(x) - \beta q_{pr(j)}(x),$$

   where both the coefficients of the monic polynomial

   $$c(x) = x^{k-j+1} - \sum_{i=0}^{k-j} \alpha_i x^i$$

   and β are determined by performing the scalar products $\langle q_k(x), q_{j+i}(x)\rangle$, $0 \leq i \leq k-j$;
   - Set $pr(j+i) = j$ for $1 \leq i \leq k-j$, $pr(k+1) = k$ and $j = k+1$.
4. Check if $q_n(x_j) = 0$, $0 \leq j \leq N$, that is, check whether or not the data set is a degenerate configuration.

By applying Algorithm 2 to the data set (x_i, f_i^{-1}), we may compute the coefficients of the polynomial $p_m(x)$, $m = N - n$, provided that $\det H'_{m-1} \neq 0$ or, equivalently, $\det H_{N-m} \neq 0$ in view of Proposition 5. In this way, we arrive at the following proposition.

PROPOSITION 7. *Given the matrix H_N, let $0 = m_0 < m_1 < \cdots < m_L = N+1$ be integers such that $\det H_{m_i-1} \neq 0$, $i = 1, \ldots, L$, otherwise $\det H_k = 0$. Then we may compute all rational interpolants $r_{N-m_i,m_i}(x) = p_{N-m_i}(x)/q_{m_i}(x)$ using $O(N^2)$ arithmetic operations.*

PROOF. At Step 3 of Algorithm 2, in view of the Hankel structure of the matrices (13), we need to compute only $\delta_i = m_i - m_{i-1}$ scalar products, thus requiring $\delta_i N$ arithmetic operations.

Proposition 7 ensures that Algorithm 2 realizes the optimal sequential solution of the rational interpolation problem without no restrictions are imposed.

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